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# Sequential Optimization Approach for the Multi-Class User Equilibrium Problem in a Continuous Transportation System



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We consider a city region with several facilities that are competing for customers of different classes. Within the city region, the road network is dense, and can be represented as a continuum. Customers are continuously distributed over space, and they choose a facility by considering both the transportation cost and market externalities. More importantly, the model takes into account the different transportation cost functions and market externalities to which different customer classes are subjected. A logit-type distribution of demand is specified to model the decision-making process of users' facility choice. We develop a sequential optimization approach to decompose the complex multi-class and multi-facility problem into a series of smaller single-class and single-facility sub-problems. An efficient solution algorithm is then proposed to solve the resultant problem. A numerical example is given to demonstrate the effectiveness and potential applicability of the proposed methodology.

## Introduction

Multiple user classes usually exist in a real transportation system. Different classes of users are likely to perceive costs differently, be required to pay different charges, and have different origin-destination (O-D) patterns. Hence, it is desirable for researchers to take this feature

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into account when modeling a transportation system. With the introduction of multiple user classes, the network equilibrium model has been applied to closely reproduce the real-life situation and generate realistic and sensible forecasts of traffic flows and market shares. Defermos (1972) was one of the first to formulate the multi-class network equilibrium problem. She considered the multi-class problem for a discrete network with links that had different cost functions for different classes of users. Recently, Vliet et al. (1986) extended the work for more general cost functions, and Lam and Huang (1992) considered a combined distribution and assignment problem. More recently, Wong et al. (2003) studied a combined distribution, hierarchical mode choice, and assignment network model with multiple user and mode classes, and Xu and Lam (2003) formulated a multi-mode network model with elastic demand.

All of the above methodological refinements were based on the discrete network modeling approach. An alternative but promising approach is the use of continuous modeling, in which the dense transportation network is approximated as a continuum. Within this modeling paradigm, it is worthwhile exploring the case of multi-class users in a continuous transportation system. This approach is very useful for macroscopic modeling of a dense transportation network in a large area, in which the detailed network information is not available or the set up of a discrete network is too demanding. With the extension of multiple user classes in the continuum model, many real-life applications can be considered; for instance, the study of the market shares of competing airports within a multi-airport region with multi-class passengers can be conducted. The explicit consideration of demand distribution is also promising, as it can reflect the facility choice of users in a transportation system. Wong and Sun (2001) incorporated the demand distribution in a continuous user equilibrium problem with competitive facilities.

We further extend the above work to the case of a combined distribution and assignment problem with multiple user classes and elastic demand. However, the problem size is significantly increased when compared to previous continuous models such as that of Wong and Yang (1999). This is because the number of variables increases dramatically and in proportion to the number of user classes and the number of facilities concerned. Under such circumstances, the conventional optimization approach for a combined problem may not be efficient. Therefore, we develop a sequential optimization approach to decompose the large combined problem into a series of smaller sub-problems. Each sub-problem becomes a single-class and single-facility problem that can be efficiently solved by a conventional solution algorithm.

This paper is organized as follows. The definitions and notation are introduced in Section 2. In Section 3, the problem formulation and solution algorithm are discussed. In Section 4, a numerical example is given to demonstrate the effectiveness and potential applicability of this sequential optimization approach for the multi-class and multi-facility problem.

## Definitions and Notation

Figure 1 shows the modeled city region in this study. There are several facilities, which are sufficiently compact when compared with the whole region. Different classes of users are continuously distributed over space, and they are free to choose any facility based on the cost incurred when patronizing that facility. Users, which may be passengers or freight, move to their chosen facility along an optimal route by minimizing a transportation cost that takes into consideration the effect of traffic congestion. Let the number of user classes be $M$, the region of the study area be $\Omega$, and the boundary be $\Gamma$. The locations of the facilities are $O_n$, $n = 1, 2, \ldots, N$, where $N$ is the number of facilities in the study area. To avoid singularity at the facilities, it is assumed that each of the facilities is of finite size and enclosed by a boundary, $\Gamma_n$.

The following notation is used throughout the paper.

$\mathrm{f}_{mn}(x,y)$ = the flow vector at location $(x,y)$ for class $m$ users heading for facility $n$

$\left|\mathbf{f}_{mn}(x,y)\right|$ = the norm of flow vector at location $(x,y)$ for class $m$ users heading for facility $n$

$f_{mnx}(x,y)$ = $x$ component of the flow vector at location $(x,y)$ for user class $m$ heading for facility $n$

$f_{mny}(x,y)$ = $y$ component of the flow vector at location $(x,y)$ for user class $m$ heading for facility $n$

$c_m(x,y)$ = the transportation cost per unit distance of movement at location $(x,y)$ for class $m$ users

$a_m(x,y)$ = the free-flow transportation cost per unit distance of movement at location $(x,y)$ for class $m$ users

$b_m(x,y)$ = the sensitivity parameter at location $(x,y)$, which represents the congestion effect perceived by class $m$ users

$C_{mn}$ = the facility cost perceived by class $m$ users at facility $n$
$u_{mn}(x,y)$ = the patronization cost (including transportation and facility costs) potential at location $(x,y)$ for class $m$ users who patronize facility $n$
$\bar{u}_m(x,y)$ = the log-sum patronization cost (including transportation and facility costs) potential at location $(x,y)$ for class $m$ users who patronize all facilities
$q_m(x,y)$ = the demand rate at location $(x,y)$ of class $m$ users
$q_{mn}(x,y)$ = the demand rate at location $(x,y)$ of class $m$ users who patronize facility $n$
$Q_{mn}$ = the demand of class $m$ users who patronize facility $n$
$\zeta_m(x,y)$ = the parameter in the logit distribution function at location $(x,y)$ for class $m$ users

The local transportation cost, which depends on location and the flow intensities at a particular location, is defined as:

$$c_m = a_m + b_m \sum_{s=1}^{N} \sum_{r=1}^{M} \left| \mathbf{f}_{rs} \right|, \quad \forall (x,y) \in \Omega . \tag{1}$$

This is an isotopic cost function because it depends on the flow intensity but not on the flow direction. The second term of equation (1) represents the congestion component as perceived by class $m$ users. This also includes the flow intensities of other classes because there are interactions among different classes of users at a location in the transportation system. The demand rate is defined as the number of users per unit area. For each class of users, the demand rate depends on the log-sum patronization cost, which includes the transportation and facility costs,

$$q_m = D_m(\bar{u}_m), \tag{2}$$

where $D_m(.)$ is a strictly decreasing demand function of class $m$ users, and $\bar{u}_m$ is the log-sum equivalent cost of class $m$ users patronizing all facilities,

$$\bar{u}_m = \frac{-1}{\zeta_m} \ln \sum_{s=1}^{N} e^{-\zeta_m u_{ms}}, \quad \forall (x,y) \in \Omega . \tag{3}$$

The users are distributed to different facilities based on the following logit-type distribution function:

$$q_{mn} = q_m \frac{e^{-\zeta_m u_{mn}}}{\sum_{s=1}^{N} e^{-\zeta_m u_{ms}}}, \quad \forall (x,y) \in \Omega . \tag{4}$$

If there is a sub-region in which the demand does not exist, then we can set $D_m(\overline{u}_m) = 0$ for all $(x,y)$ in this sub-region, so that the demand rate vanishes regardless of the patronization cost incurred. Equation (4) distributes the demand to different facilities in accordance with the multinomial logit form, and equation (3) specifies the corresponding log-sum patronization cost which represents the aggregated patronization cost over all of the facilities in the multinomial logit context (Ortuzar and Willumsen, 1994). When the patronization cost to a particular facility, $u_{mn}$, increases, less demand is attracted to this more expensive facility. For each combination of user class and facility, the demand rate and flow vector must satisfy the flow conservation equation:

$$\nabla \mathbf{f}_{mn} + q_{mn}, \quad \forall (x,y) \in \Omega = 0 . \tag{5}$$

Assuming that there is no flow across the boundary of the study area, we have the following boundary condition:

$$\mathbf{f}_{mn} = 0, \quad \forall (x,y) \in \Gamma . \tag{6}$$

However, it is easy to extend the model to represent a given entry or exit demand pattern across the boundary by replacing the above constraint with $\mathbf{f}_{mn} \cdot \mathbf{n} = g_{mn}(x,y)$, where $g_{mn}$ is the entry or exit demand. The number of class $m$ users who are attracted to facility $n$ is obtained as $Q_{mn} = \iint_{\Omega} q_{mn} \mathrm{d}\Omega$. From the flow conservation condition around facility $n$, the demand can also be determined by:

$$Q_{mn} + \int_{\Gamma_n} \mathbf{f}_{mn} \cdot \mathbf{n} \mathrm{d}\Gamma = 0 . \tag{7}$$

## Sequential Optimization Approach

In this section, a sequential optimization approach is formulated to ensure that the solution of the problem satisfies all of the functional requirements that were given in Section 2. We first introduce a single-class and single-facility sub-problem as follows.

Single-class and single-facility sub-problem

The single-class and single-facility sub-problem takes a diagonalization approach, in which all of the variables that are related to other classes and facilities are held fixed. To form the optimization problem, the cost and demand functions are modified as follows. For the cost function in equation (1), the modified cost function for the sub-program of class $m$ to facility $n$ combination is given as:

$$c_{mn}=a_m+b_m\sum_{s=1}^{N}\sum_{r=1}^{M}|\mathbf{f}_{rs}|=a_m+b_m\sum_{s=1}^{N}\sum_{r=1,(r,s)\neq(m,n)}^{M}|\mathbf{f}_{rs}|+b_m|\mathbf{f}_{mn}|=a_{mn}+b_m|\mathbf{f}_{mn}|, \qquad (8)$$

where $a_{mn}=a_m+b_m\sum_{s=1}^{N}\sum_{r=1,(r,s)\neq(m,n)}^{M}|\mathbf{f}_{rs}|$ is assumed constant in the above sub-problem. Therefore, the flows from other classes are considered as background flows, the congestion effects of which are added into the constant term of the modified cost function.

For the demand function of class $m$, the relationship between $q_{mn}$ and $u_{mn}$ is obtained as follows. Substituting equations (2) and (3) into equation (4), we have

$$q_{mn}=D_m\left(\frac{-1}{\zeta_m}\ln\sum_{s=1}^{N}e^{-\zeta_m u_{ms}}\right)\frac{e^{-\zeta_m u_{mn}}}{\sum_{s=1}^{N}e^{-\zeta_m u_{ms}}}=D_{mn}(u_{mn}), \quad \forall(x,y)\in\Omega, \qquad (9)$$

while keeping all other variables $u_{rs}, r\neq m, s\neq n$, unchanged. The optimization problem for the sub-problem of class $m$ to facility $n$ combination can be set up as follows:

$$\underset{\mathbf{f}_{mn}}{\text{Minimize}}$$

$$z_{mn}(\mathbf{f}_{mn})=C_{mn}Q_{mn}+\iint_{\Omega}a_{mn}|\mathbf{f}_{mn}|+\frac{1}{2}b_m|\mathbf{f}_{mn}|^2-\int_0^{q_{mn}}D_{mn}^{-1}(\xi)\,\mathrm{d}\xi\,\mathrm{d}\Omega \qquad (10a)$$

subject to $\nabla\mathbf{f}_{mn}+q_{mn}=0, \qquad (10b)$

$$\mathbf{f}_{mn} = 0, \tag{10c}$$

$$Q_{mn} + \int_{\Gamma_{nc}} \mathbf{f}_{mn} \cdot \mathbf{n} d\Gamma = 0, \tag{10d}$$

where the function $D_{mn}(u_{mn})$ is defined in equation (9). The solution of the sub-program (10) can be obtained by an efficient finite element method formulated by Wong and Yang (1999). In their work, the above minimization problem was proven to satisfy the user equilibrium conditions. Interested readers are referred to the paper for more details.

Fixed point problem

Let the solution of the multi-class and multi-facility problem be $\Psi$, and the solution of the minimization problem (10) be $\psi_{mn}$. Denote $\Psi^*$ as the solution at the current iteration. The minimization problem (10) can be summarized in the following abstract form:

$$\psi_{mn} = F_{mn}\left(\Psi^*\right). \tag{11}$$

We then form an updated solution vector by replacing the corresponding elements in $\Psi^*$ with $\psi_{mn}$. Equation (11) can be revised as:

$$\Psi_{mn} = \widetilde{F}_{mn}\left(\Psi^*\right). \tag{12}$$

By sequential optimization and updating for all user classes and all facilities, we have

$$\Psi_{MN} = \widetilde{F}_{MN}\left(\cdots\left(\widetilde{F}_{M1}\left(\cdots\left(\widetilde{F}_{1N}\left(\cdots\left(\widetilde{F}_{12}\left(\widetilde{F}_{11}\left(\Psi^*\right)\right)\right)\right)\right)\right)\right)\right) = G\left(\Psi^*\right). \tag{13}$$

If we can find a mutually consistent $\Psi^*$ such that $\Psi^* = G\left(\Psi^*\right)$, then $\Psi^*$ is said to be a fixed point solution that satisfies all of the functional relationships and user equilibrium conditions of all sub-problems. Hence, $\Psi^*$ becomes the solution of the combined multi-class and multi-facility problem.

Solution procedure

The finite element method is used to solve the sub-problem (Wong and Yang, 1999). However, the solution procedure for the fixed point problem of the multi-class-multi-facility situation is still missing in the literature on continuous equilibrium modeling. We derive and work out the following solution procedure.

Step 1: Find an initial solution for $\Psi^{(1)}$. Set $k = 1$.
Step 2: Set m = 1 and n = 1.
Step 3: Solve sub-problem (m, n). Update the solution vector $\Psi_{mn}^{(k)}$.
Step 4: If n < N, then set n = n + 1 and go to Step 3. Otherwise, go to Step 5.
Step 5: If m < M, then set m = m + 1 and go to Step 3. Otherwise, go to Step 6.
Step 6: Evaluate $\varepsilon = \left\| \Psi^{(k)} - \Psi_{MN}^{(k)} \right\|$.
Step 7: If ε is less than an acceptable error, then stop and $\Psi^{(k)}$ is the solution.
Step 8: Otherwise, set $\Psi^{(k+1)} = \Psi_{MN}^{(k)}$ and k = k + 1. Go to Step 2.

In the procedure, the error term $\left\| \Psi^{(k)} - \Psi_{MN}^{(k)} \right\|$ is chosen as the stop criterion of the fixed point problem, which ensures that for the solution vector, Ψ, each of the minimization problems gives a solution that is sufficiently close to the final solution. Moreover, the solution vector Ψ is updated by the sequential solution of each sub-problem, by which a better convergence can be achieved.

**Numerical Example**

In this section, we present a numerical example to illustrate the effectiveness and potential applicability of the sequential optimization approach for solving the user equilibrium problem of a multi-class and multi-facility within a continuous system. To exemplify the potential applicability of the model, we assume that the modeled region has an export-oriented economy and that it takes the form shown in Figure 1. There are three facilities in the region, and two classes of users are considered to make their facility choices within the system. The three facilities are considered to be port facilities and the two classes of users are manufacturers of light high-value-added exports and heavy low-value-added exports. We denote the former as Class 1 users and the latter as Class 2 users. The transportation cost functions of these two classes of users are:

$$\text{Class 1: } c_1 = 0.010 + 0.40 \times 10^{-4} \sum_{s=1}^{3} \sum_{r=1}^{2} \left| \mathbf{f}_{rs} \right|$$

$$\text{Class 2: } c_2 = 0.015 + 0.35 \times 10^{-4} \sum_{s=1}^{3} \sum_{r=1}^{2} \left| \mathbf{f}_{rs} \right|$$

throughout the study area, where $c_1$ and $c_2$ are measured in dollars per cubic meter per kilometer. We see that Class 1 users have a lower free-flow transportation cost (lower $a_1$) because these export commodities are lighter and often well packed in standardized cartons for easy transportation. For the transportation company, less labor needs to be deployed in the loading and unloading activities. However, Class 1 commodities are more sensitive to congestion (higher $b_1$) because high-value-added goods, such as electronic components, are often time-critical commodities. In other words, delivery time is critical to the manufacturers. Class 2 users have a higher free-flow transportation cost (higher $a_2$) because these commodities are heavier and more bulky. For the transportation company, the loading and unloading of Class 2 freight requires more labor, and sometimes special equipment or vehicle types. Yet, these commodities are less sensitive to congestion delays (lower $b_2$). Very often, they are categorized as time-definite commodities.

In this study, we assume an exponential form of the demand function in equation (2),

$$q_m = D_m(\bar{u}_m) = K_m e^{-\alpha_m \bar{u}_m}, \tag{14}$$

where $K_m$ is a potential demand rate. For the demand function of class $m$, the function $D_{mn}(u_{mn})$ that is defined in equation (9) can be obtained as follows. Taking logarithm on both sides of equation (14), we have

$$\ln D_m(\bar{u}_m) = \ln K_m - \alpha_m \bar{u}_m. \tag{15}$$

Substituting equation (3) into equation (15), we have

$$\ln D_m(\mathbf{u}) = \ln K_m + \frac{\alpha_m}{\zeta_m} \ln \sum_{s=1}^{N} e^{-\zeta_m u_{ms}}, \tag{16a}$$

or

$$D_m(\mathbf{u}) = K_m \left( \sum_{s=1}^{N} e^{-\zeta_m u_{ms}} \right)^{\frac{\alpha_m}{\zeta_m}}, \tag{16b}$$

where u is a vector that contains all $u_{mn}$. However, for the sub-program of the combination of class $m$ and facility $n$, $S_{mn} = \sum_{s=1, s \neq n}^{N} e^{-\zeta_m u_{ms}}$ is constant. Equation (16b) can be re-written as:

$$D_m(u_{mn}) = K_m \left( S_{mn} + e^{-\zeta_m u_{mn}} \right)^{\frac{\alpha_m}{\zeta_m}}. \tag{17}$$

Substituting equation (17) into equation (4), we obtain the equivalent demand function for the sub-problem of class $m$ and facility $n$ combination,

$$D_{mn}(u_{mn}) = K_m \left( S_{mn} + e^{-\zeta_m u_{mn}} \right)^{\frac{\alpha_m}{\zeta_m} - 1} e^{-\zeta_m u_{mn}}. \tag{18}$$

The demand functions of the two classes are considered to be different:

Class 1: $q_1 = 450 e^{-0.30 \bar{u}_1}$

Class 2: $q_2 = 650 e^{-0.25 \bar{u}_2}$

where $q_1$ and $q_2$ are measured in $m^3$ per square kilometer of land area within the study region. We assume that Class 1 users have a more elastic demand. This is likely because light high-value-added exports can readily shift to other modes, such as air, or use other port facilities outside of the region should there be port delays or should the total generalized cost of the port facilities in the region be considered non-competitive. Class 2 users are assumed to have a less elastic demand. The higher per unit distance transportation cost of these heavy and bulky exports will tend to force Class 2 users to use nearby port facilities.

Moreover, these low-value-added exports cannot afford a modal shift to air transportation.

In freight transportation, a deterministic distribution function based on the absolute lowest total transportation and facility cost is often not applicable (Regan and Golob, 2000). Instead, a logit-type distribution function can be used to represent the distribution of users among different port facilities (equation 4). The parameter $\zeta_m$ that is used in the logit distribution function is taken to be 0.6 and 0.7 for Class 1 and Class 2 users respectively. As this numerical example is a hypothetical case, there are no strong reasons for the parameters to take the value of 0.6 or 0.7, or any other specific value. Basically, we take the parameter for Class 1 users to be smaller because these users have higher flexibility in transporting their commodities and they are more likely to consider non-monetary facility attributes, such as security, which may not be fully captured in the total cost estimation. When the model is applied to solve a real-life problem, all parameters and functions need to be carefully calibrated based on actual data.

Lastly, the model takes into account that port facilities 1, 2, and 3 offer different freight rates to Class 1 and Class 2 users. Once again, the use of different freight rates for commodities of different types is common (Matear and Gray, 1993). In our model, this is captured by the different facility cost that is charged to the two classes of users at the port facilities. To illustrate, we assume that the facility cost is measured in freight rates per $m^3$. For Class 1 users, the facility cost is assumed to be \$2.2, \$2.1, and \$2.4 per $m^3$ for Ports 1, 2, and 3 respectively. For Class 2 users, we assume that the facility cost to be \$2.5, \$1.8, and \$2.3 per $m^3$ for Ports 1, 2, and 3 respectively. Once again, these data are hypothetical and only aim to test and illustrate the effectiveness and potential applicability of the model.

With the above numerical definition of the cost and demand functions, the example of a multi-class and multi-facility problem is set up and solved by means of the methods discussed in Section 3. Figure 2 shows the convergence of the solution algorithm. The solution converges satisfactorily in 30 iterations. The graphical results of the flow pattern, flow intensity, transportation cost, and probability of Class 1 users using Port 1, and the associated demand distribution over space are shown in Figures 3 to 7 respectively. Similarly, the graphical results for Class 1 users who patronize the other port facilities and for Class 2 users who use the three port facilities can be generated. The market shares of the three Ports in this numerical example are summarized in the Table 1.

**Table 1** Distribution of demand to different port facilities

| | Class 1 Users | | Class 2 Users | |
|---|---|---|---|---|
| | Usage ($m^3$/hr) | Market Share (%) | Usage ($m^3$/hr) | Market Share (%) |
| **Port 1** | 36890 | 35.3 | 51576 | 28.8 |
| **Port 2** | 35715 | 34.2 | 72300 | 40.4 |
| **Port 3** | 31921 | 30.5 | 55062 | 30.8 |
| **Total** | 104526 | 100.0 | 178938 | 100.0 |

Some of the important results and interpretations of the continuum network modeling such as the tracing of paths taken from the flow pattern, the concentration of users around the port facilities, etc, will not be discussed in this paper. Interested readers can refer to those papers on continuum modeling. Our key concern is to show the effectiveness of the proposed sequential approach in giving a reasonable and feasible solution to the multi-class and multi-facility user equilibrium problem. This effectiveness can be demonstrated in the ways in which different classes of users interact and affect each other, and thus result in the equilibrium spatial traffic flow patterns and market share distribution. Different classes of users interact because users (of both classes) generate market externalities. To illustrate, the optimal route choice and facility choice of a Class 1 user are affected not only by the decisions of other Class 1 users but also those of Class 2 users in the system. In particular, the levels of traffic and port congestion that a Class 1 user must face cannot be explained by the behavior of users of the same class only. The ability to handle interactions among different user classes is central to effectively solving the multi-user and multi-commodity problem. Thus, we will primarily focus on discussing the results related to this point. In this example, the interaction of different classes of users can be shown clearly in the following two ways.

First, from the results of the demand distribution of Class 2 users in Table 1, we see that Port 2 has a larger market share (40.4%) than those of Ports 1 (28.8%) and 3 (30.8%). This is because the cost for Port 2 (\$1.8 per $m^3$) is much lower than those of Ports 1 (\$2.5 per $m^3$) and 3 (\$2.3 per $m^3$). Yet, for the market share distribution of Class 1 users, the story is completely different. For Class 1 users, the market share of Port 2 (34.2%) is lower than that of Port 1 (35.3%), although the facility cost at the former (\$2.1 per $m^3$) is actually lower than that of the latter (\$2.2 per $m^3$). The reason of having such "abnormal" distribution is due to the "interference" from Class 2 users. Port 2, due to its relatively low cost, obtains a huge demand from Class 2 users. As a result, a large volume of Class 2 freight is concentrated around it. The traffic and port congestion that are generated will result in dramatically higher total cost for Class 1 users to patronize Port 2. The effects thus override the comparative lower charge of Port 2 for Class 1 users and cause fewer Class 1 users to choose that port facility. In this example, we can clearly see that the facility choice of users in the transportation system is affected by the interactions between the different classes of users.

Second, we see from Figure 4 that the flow intensity of Class 1 users who patronize Port 1 is relatively high near the boundary of the study region, but it is relatively low in the vicinity of Ports 2 and 3. This is because the total demand (especially from Class 2 users) is very high (see Table 1) in Ports 2 and 3, and therefore the effect of congestion is severe in the vicinity of these two facilities. As a result, most Class 1 users who use Port 1 and are situated in the eastern part of the region will try to minimize their total cost by taking a longer but less congested path along the continuum. These detours cause the flow intensity of Class 1 users, who patronize Port 1, to be high near the boundary of the study region, $\Gamma$. In this numerical example, we also see that the optimal route choice of users in the transportation system is affected by the interactions between the two different classes of users.

## Conclusions

We have introduced a sequential optimization approach for solving the multi-class user equilibrium problem in a continuum network with multiple competitive facilities. The main aim of this method is to break down the complex multi-class problem and solve it separately for each class and each destination. A numerical example is used to demonstrate the effectiveness and potential applicability of the proposed approach. The interactions among users are shown to have affected both the optimal route choice and the facility choice of the users in the system.

The proposed methodology is particularly useful for the land-use and transportation problems at the regional level (Boyce and Mattsson, 1999; Wong et al., 1999; Haghani et al., 2003; Lee et al., 2003), and pedestrian flow problems (Lam et al., 2003; Hoogendoorn et al., 2003; Hoogendoorn and Bovy, 2004) that are an important area of research for non-motorized transportation (Khisty, 2003).

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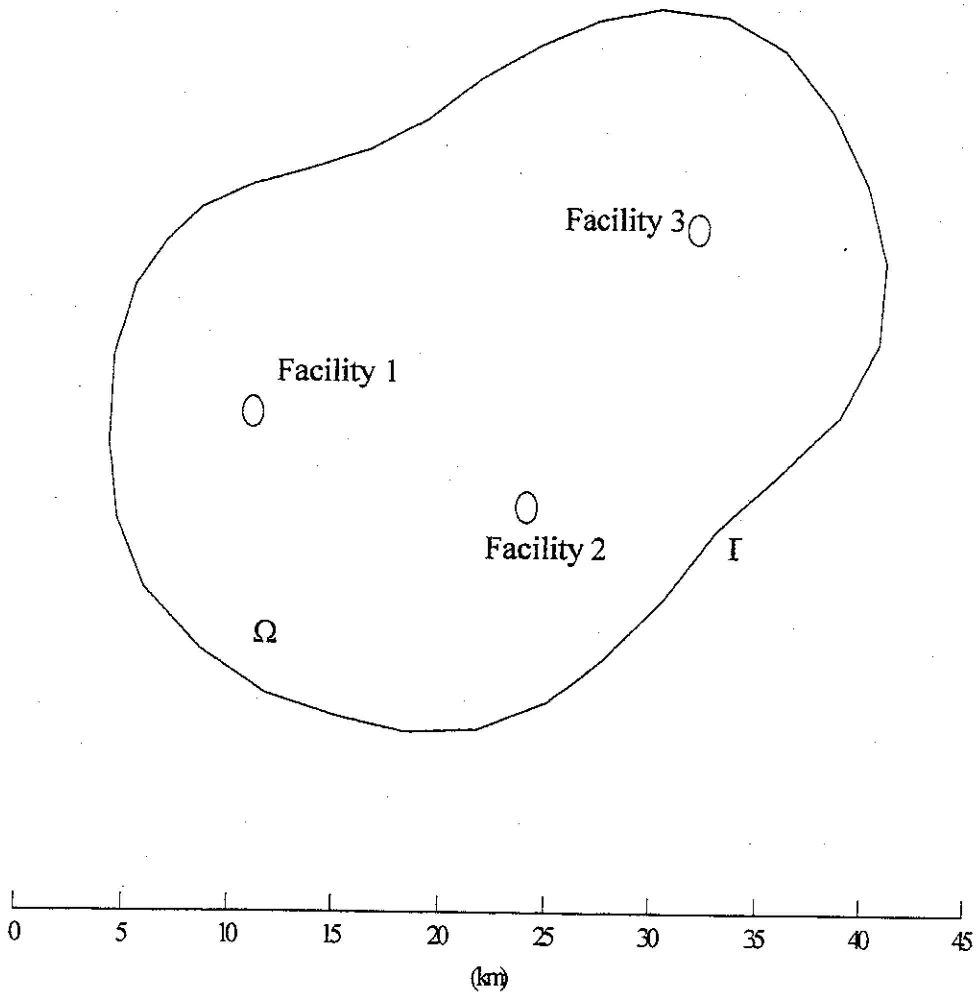


**Fig. 1** The Modelled Region

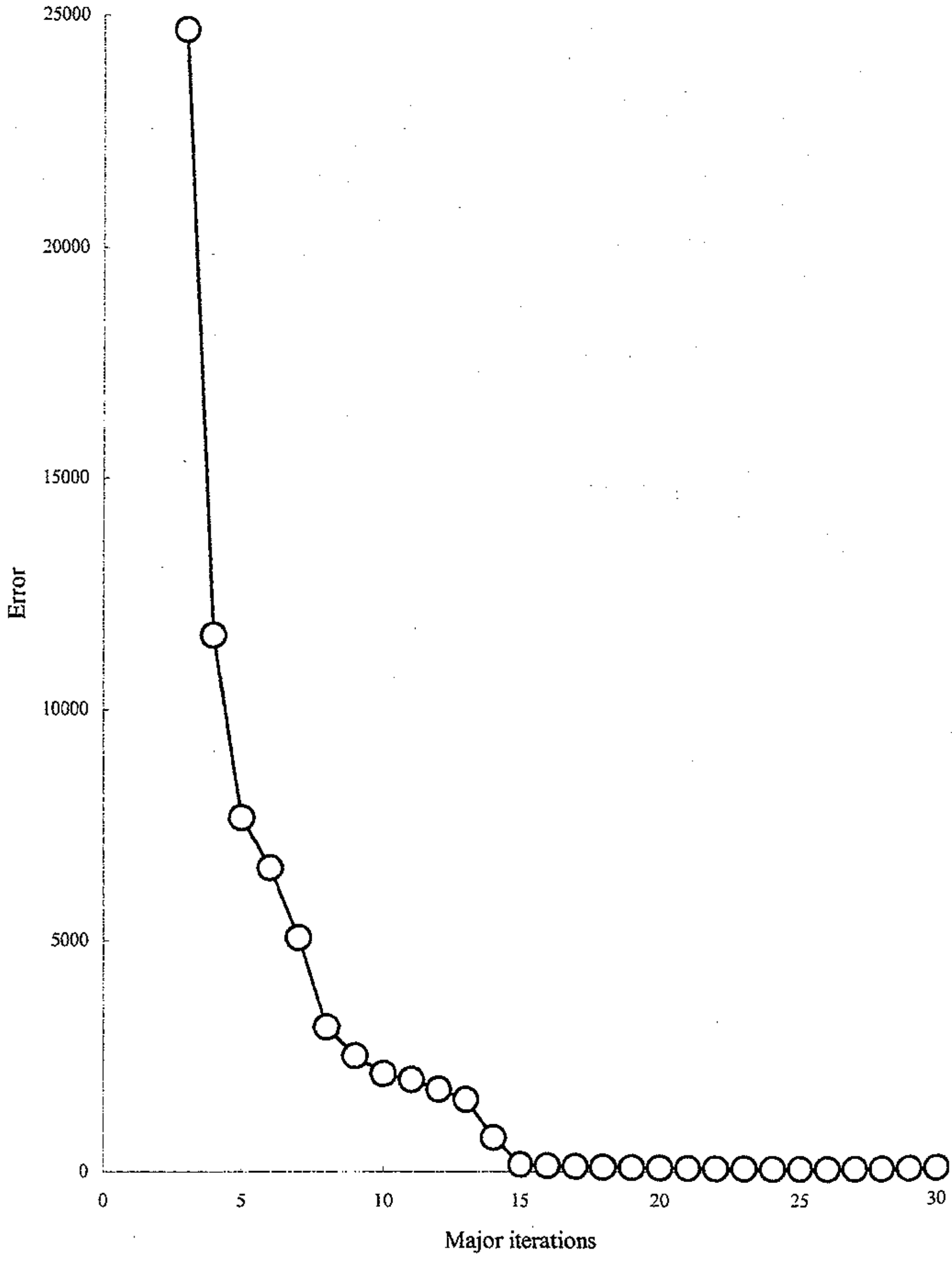


**Fig. 2** Convergence Curve

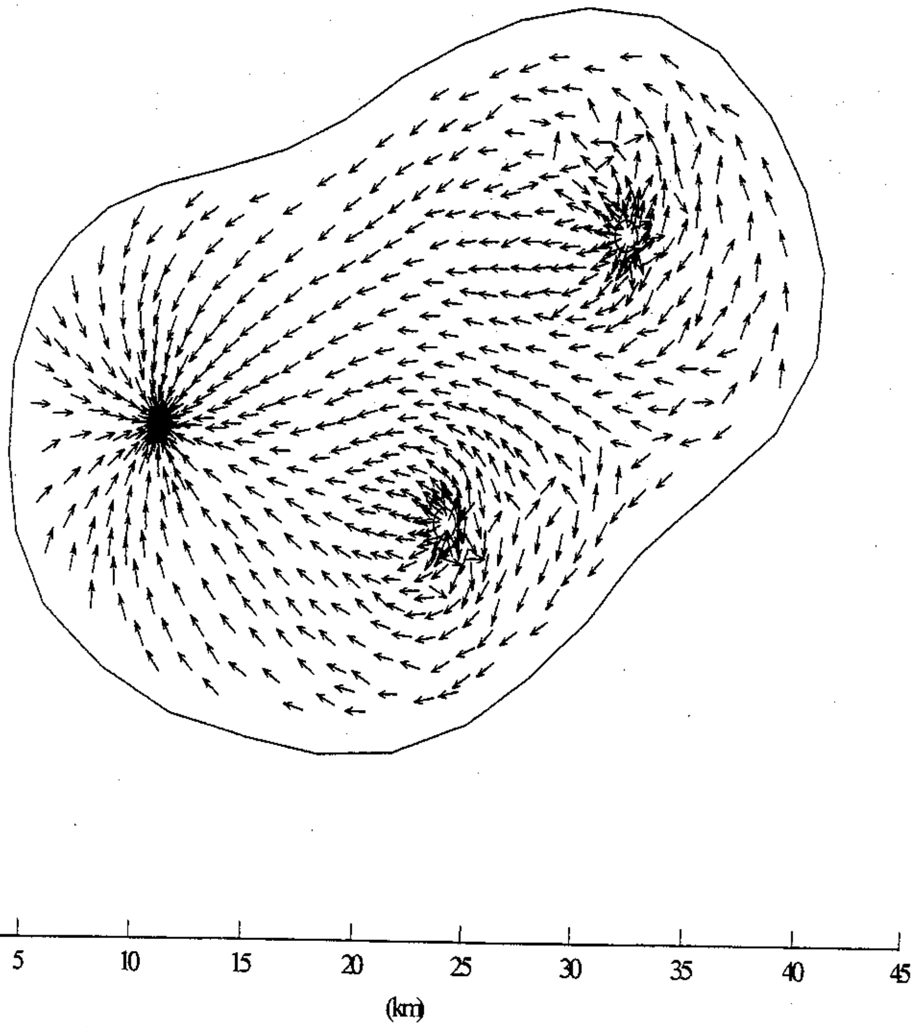


**Fig. 3** Flow pattern for Class 1 users of Facility 1

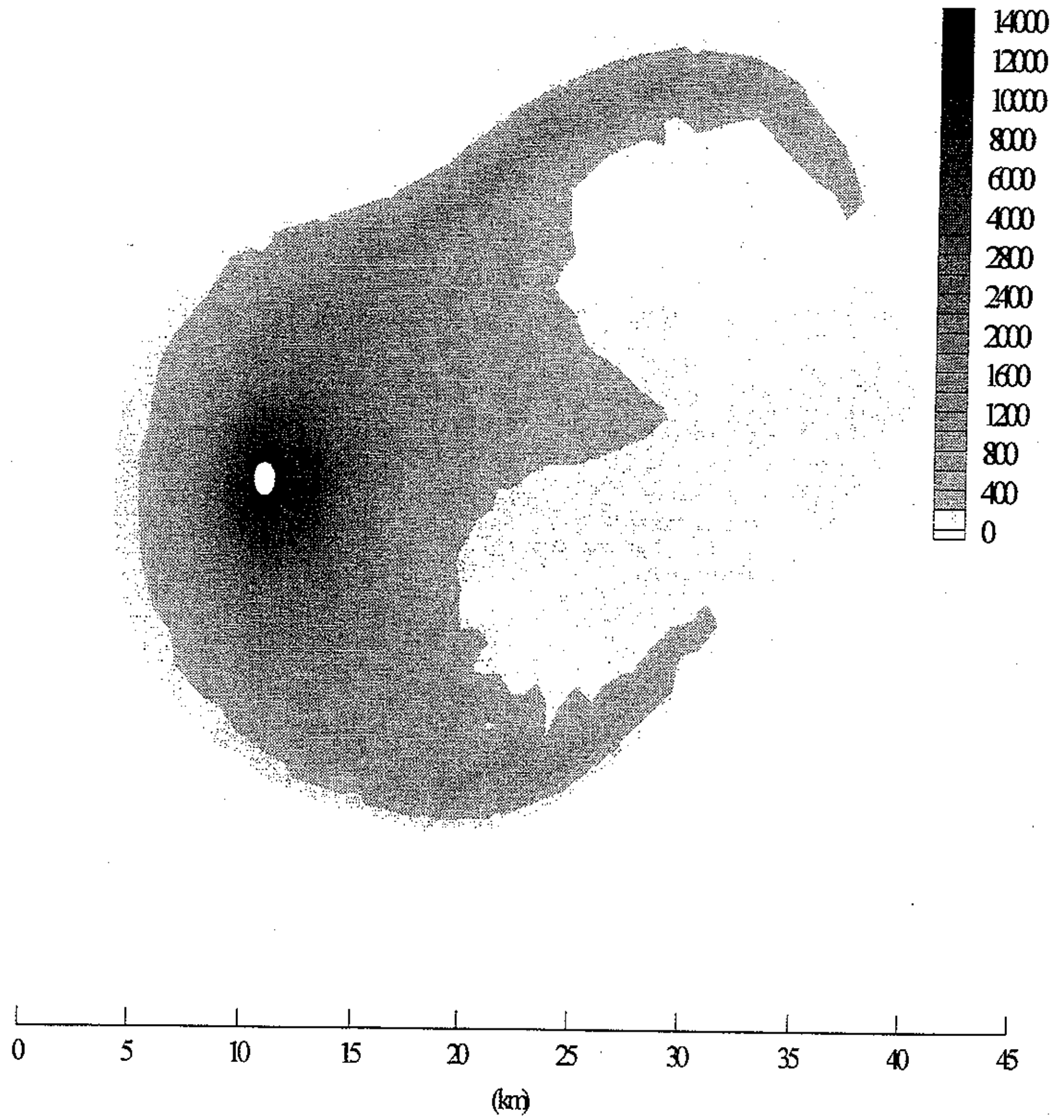


**Fig. 4** Flow intensity for Class 1 users of Facility 1 ($m^3$/km/hr)

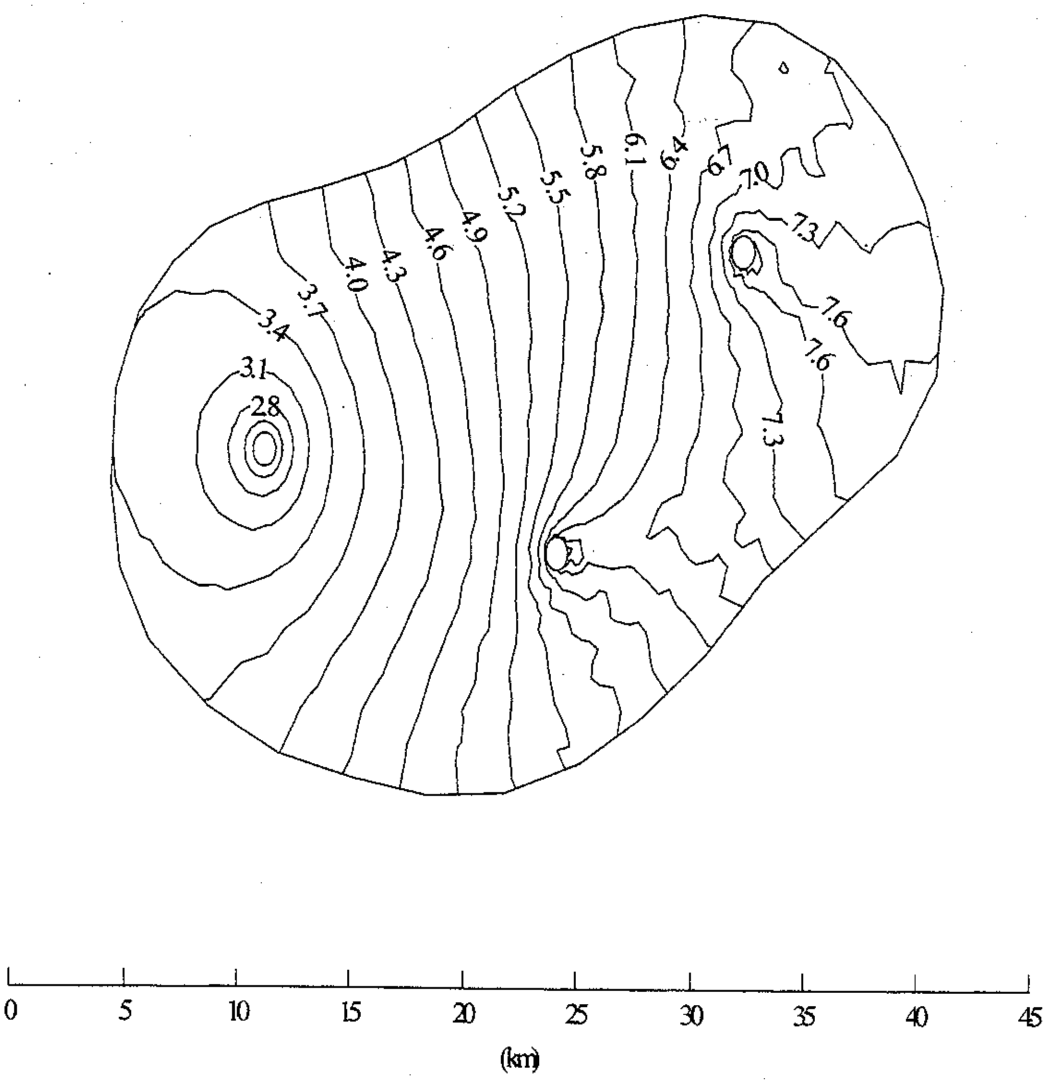


**Fig. 5** Total travel cost for Class 1 users of Facility 1 (Dollars)

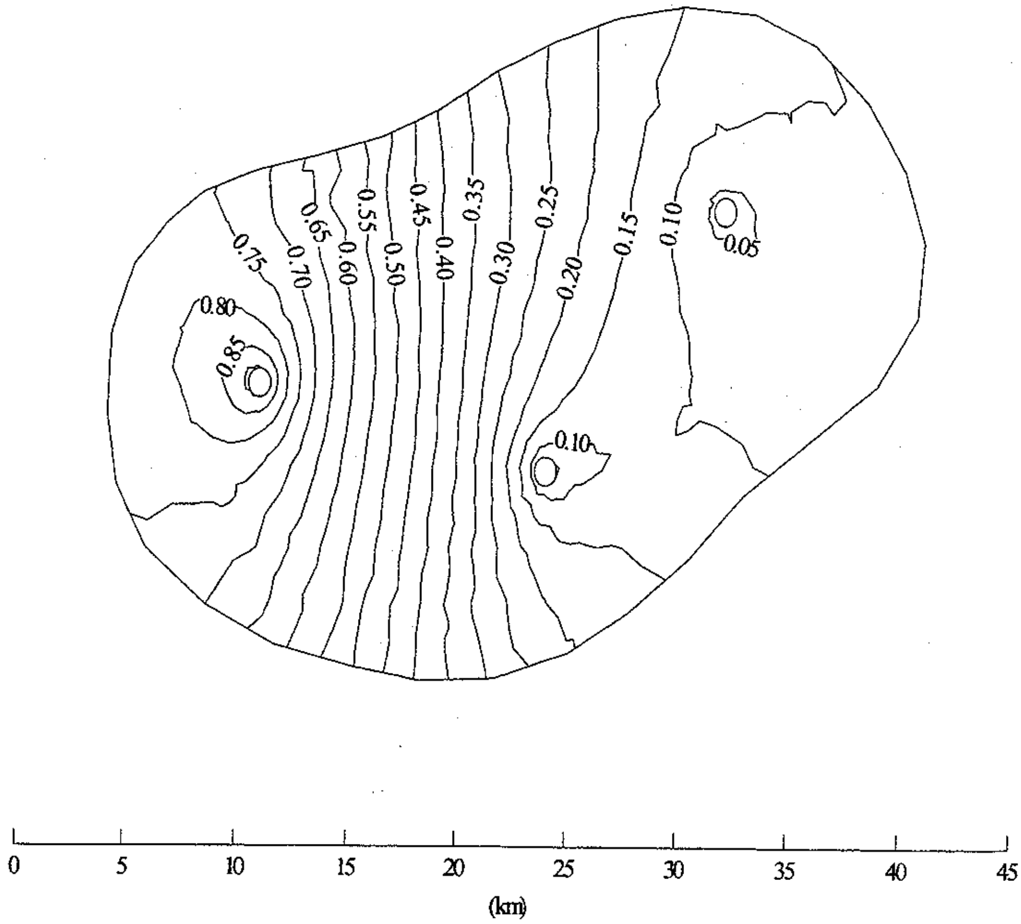


**Fig. 6** Probability of Class 1 users choosing Facility 1

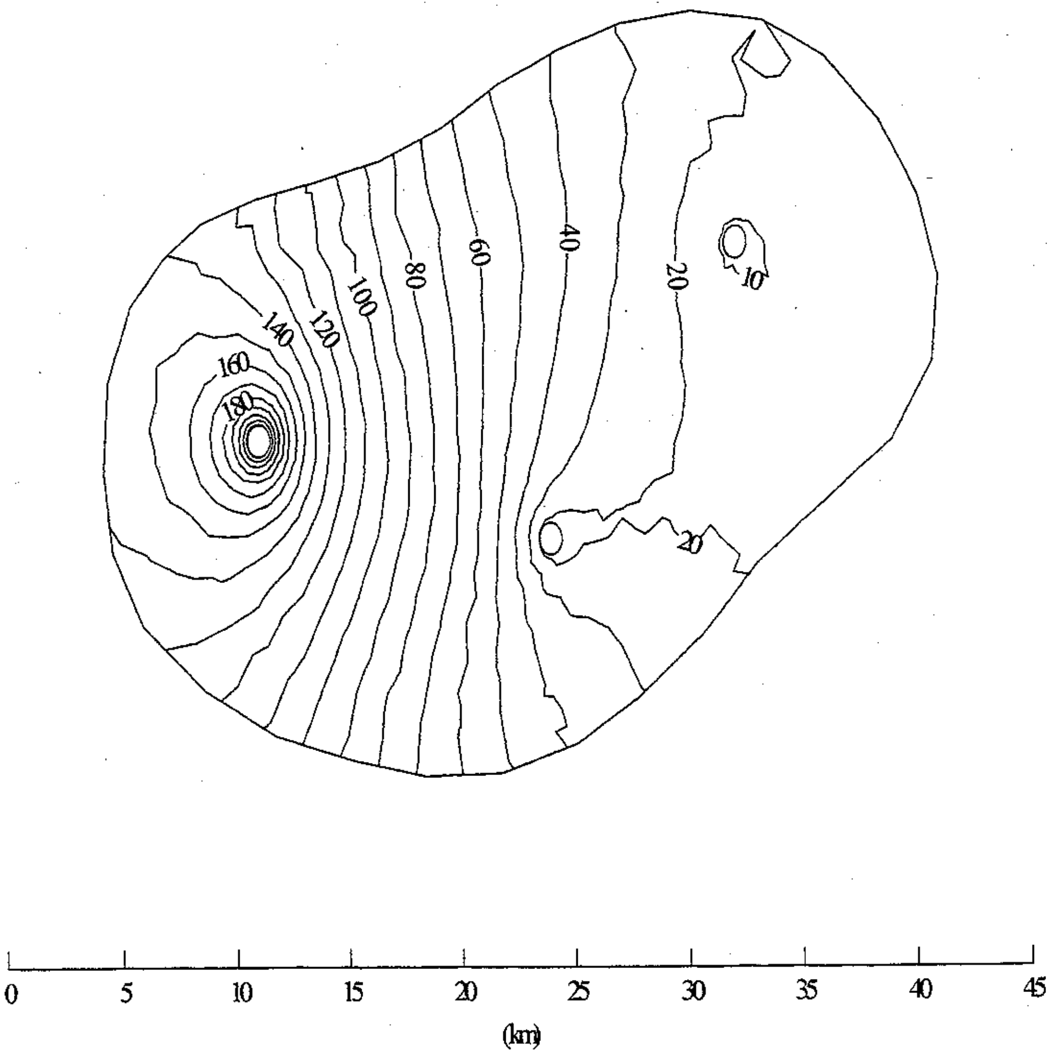


**Fig. 7** Demand of Class 1 users for Facility 1 ($m^3/km^2/hr$)