Journal de Mécanique théorique et appliquée,
Journal of theoretical and applied mechanics
Special issue, supplement n° 1 to vol. 7 - 1988

The quasistatic contact problem with friction for three-dimensional elastic bodies

J. J. KALKER

Faculty of Mathematics and Informatics, P.O. Box 356, 2600 AJ Delft, The Netherlands

Résumé - A partir de la formulation faible du problème de contact, on montre que pour de nombreux cas importants d'un point de vue technologique, tels que les contacts roue-rail et les contacts dans les roulements à bille ou à rouleaux, la surface de contact et les efforts normaux sont indépendants des forces de traction tangentielles. Des algorithmes particuliers ont été développés dans ce cas; ils sont tout à fait fiables. Un autre algorithme traitant du contact de manière plus générale est en cours de développement. Dans la fin de l'article, on discute de contacts évolutifs, et on considère le cas de roulement stationnaire.

Summary - *Starting from the weak formulation of the contact problem it is shown that in a number of technologically important cases, such as wheel-rail contact in railways, and the contacts in roller bearings, the contact formation is independent of the tangential traction. Special algorithms have been developed for this case; they are fully reliable. Another algorithm dealing with more general contacts is still under development. The paper closes with a discussion of contact evolution, and notably steady state rolling is considered.*

1. INTRODUCTION

Starting from the weak formulation of the contact problem it is shown that in a number of technologically important cases, such as wheel-rail contact in railways, and the contacts in roller bearings, the contact formation is independent of the tangential traction. This justifies a special study of this case. An algorithm for contact formation and one for friction have been developed which under those conditions lead to a solution in a finite number of steps. They have been implemented for the elastic half-space, and the computer program has never failed up to now.

An amalgamation of the contact formation and the friction algorithms has been also implemented. This method, however, does not converge in all possible cases.
Finally, contact evolutions, notably steady state rolling, are discussed.

2. STATEMENT OF THE PROBLEM

We consider two bodies labeled 1 and 2. Their particles (x_i), $i = 1,2,3$, are described Lagrangean fashion, by their coordinates in the undeformed state, in which there are no stresses acting in the bodies. The bodies in the undeformed state are denoted by V_{a0}, $a = 1,2$, where a subscript 0 refers to the undeformed state. Their surfaces are denoted by ∂V_{a0}. The bodies undergo a deformation which for a generic particle (x_i) is described by $(x_i) \to (y_i)$, where y_i are the coordinates of the particle (x_i) in the deformed state. We define the displacement (u_i) by

$$u_i = y_i - x_i \tag{1}$$

The interior and surface of body a in the deformed state are given by V_a, ∂V_a.
The deformation calls up a stress field. The Piola stress σ_{0ij} is defined as the force in i-direction per unit undeformed area with outer normal in the j-direction. The surface traction in i-direction on the undeformed geometry is called p_{0i}, and the surface traction on the deformed geometry is called p_i.

A third state is defined; it is called the reference state. It is a state with known displacement and stress, so close to the deformed state that one may linearize the deformed quantities about it. Quantities pertaining to the reference state are given the subscript r. The reference state is convenient in describing rolling: The undeformed state is immutable, but the reference state is a state with known properties that moves with the bodies.

The surfaces ∂V_{a0}, ∂V_a, ∂V_{ar} $(a = 1,2)$ are each divided into three corresponding regions, viz., ∂V_{a0u}, ∂V_{a0p}, ∂V_{a0c}, etc. In ∂V_{a0u} the displacement u_i is prescribed as $\overline{u}_i$. In ∂V_{a0p} the surface traction p_i is prescribed as $\overline{p}_i$. ∂V_{a0c}, ∂V_{arc}, ∂V_{ac} are called the potential contact area, which consist of the contact area and a neighborhood of it. The deformed potential contacts (pot.con.) and thus the reference pot.con. are so close that we may refer them to a single surface ∂V_c. The contact area with respect to this surface we call N, and B is defined as $\partial V_c \setminus N$. In the pot.con. ∂V_c distinction is made between vector components along the normal $\underline{n}$ position into body 1, which is given a subscript n, and the tangential components which are orthogonal to this normal. They are given a Greek index, which runs from 1 to 2.

In the pot.con. ∂V_c contact conditions hold, viz.

$$\left.\begin{array}{ll}
p_i = 0 \quad \text{in B,} & \text{(a)} \\
e(x_i) = \text{deformed distance at } (x_i)\in\partial V_c \text{ measured along } \underline{n}\text{, from 2 to 1;} & \\
e \geq 0, \quad e = 0 \text{ in K;} & \text{(b)} \\
p_n = p_{1n} \text{ is compressive } (\geq 0) \text{ in K;} & \text{(c)} \\
|p_\tau| \overset{\text{def}}{=} \sqrt{p_1{}^2 + p_2{}^2} \leq fp_n, \quad f\text{ : coefficient of friction; we call } g = fp_n \text{ the} & \\
\text{traction bound.} & \text{(d)} \\
s \overset{\text{def}}{=} \text{slip of body 1 over body 2;} & \\
\text{if } |s_\tau| \neq 0 \text{ then } p_\tau = -fp_n s_\tau/|s_\tau| & \text{(e)}
\end{array}\right\} \tag{2}$$

We have

$$\left.\begin{array}{l}
\text{for the deformed distance: } e(x_i) = h(x_i) + \Delta u_n \\
\text{with} \\
h(x_i) \text{ in the distance between the reference states, measured from body 2 to} \\
\text{body 1, i.e. along } \underline{n};
\end{array}\right\} \tag{3}$$

$$\Delta u_i \overset{\text{def}}{=} u_{1ri} - u_{2ri}, \quad (u_{ari})\text{: displacement of body a with respect to the reference state} \tag{4}$$

We consider increments in time from the previous instant t' to the present instant t; t' < t; quantities pertaining to the instant t' are given a prime ('). The slip of body 1 over body 2 at (x_i) follows from

$$(t-t')s_\tau(x_i) = (t-t')\dot{w}_\tau(x_i) + \Delta u_\tau - \Delta u'_\tau \ , \qquad (5)$$

where

$\dot{w}_\tau(x_i)$ is the rigid slip, i.e. the slip of reference state of body 1 over the reference state of body 2.

$\Delta u'_\tau$ is Δu_τ at the previous instant t'; $\Delta u'_\tau$ is assumed known a priori.

We can now state the problem:

Given the undeformed state and the reference state, in particular the deformation between the two and the stresses in the reference state; as well as the reference distance $h(x_i)$ and the rigid slip w_τ, find the deformed state: its displacement, and its stresses, in particular the surface displacement and the surface loads.

3. THE PRINCIPLES OF VIRTUAL AND COMPLEMENTARY VIRTUAL WORK

Virtual work:

$$0 \geq \delta Q \quad \forall \delta u_i \text{ sub } u_i = \bar{u}_i \text{ in } \partial V_{a0u} ; \quad \delta Q\text{: virtual work;}$$
$$e = h + \Delta u_n \geq 0 \text{ in } \partial V_c ;$$
$$(t-t')s_\tau \overset{def}{=} (t-t')w_\tau + \Delta u_\tau - \Delta u'_\tau \ , \qquad (6a)$$

$$\partial Q = \sum_{a=1,2} [\int_{V_{a0}} \{-\sigma_{0ij}\delta u_{i,j} + (f_{0i} - \rho_0 \ddot{u}_i)\delta u_i\} dV_0 + \int_{\delta V_{a0p}} \delta(\bar{p}_{0i}u_i) dS_0] + - \int_{\partial V_c} (t-t') g\delta |s_\tau| dS \qquad (6b)$$

$$= \sum_{a=1,2} [\int_{V_{a0}} \{\sigma_{0ij,j} + f_{0i} - \rho_0 \ddot{u}_i\} \delta u_i dV_0 + \int_{\partial V_{a0p}} (\bar{p}_{0i} - p_{0i}) \delta u_i dS_0 + - \int_{\partial V_c} \{p_n \delta(\Delta u_n) + p_\tau \delta(\Delta u_\tau) + (t-t') g\delta |s_\tau|\} dS \qquad (6c)$$

where summation over repeated indices is assumed; the notation $q_{i,j}$ signifies: $\partial q_i/\partial x_j$;

δq: variation of q.

(f_{0i}) is the body force per unit undeformed volume;

$\cdot \overset{def}{=} \frac{d}{dt}$, material derivative; ρ_0: density in undeformed state; (6d)

the notation $|q_i|$ signifies $\sqrt{q_1^2 + q_2^2 + q_3^2}$, and $|q_\tau| = \sqrt{q_1^2 + q_2^2}$.

Complementary virtual work, defined only when $\rho_0 = 0$ (inertialess, or quasistatic problem):

$\delta C \geq 0 \quad \forall \ \sigma_{ij}, p_i$ sub $\rho_0 = 0$ (restriction), and $\sigma_{0ij,j} + f_{0i} = 0$ in V_{a0} (equilibrium)

δC: complementary virtual work; $p_i = \bar{p}_i$ in ∂V_{a0p}

$$p_n \geq 0, \quad |p_\tau| \leq g = f p_n \text{ in } \partial V_c \qquad (7a)$$

$$\delta C = \sum_{a=1,2} \{ \int_{V_{a0}} u_{i,j} \delta\sigma_{0ij} dV_0 - \int_{\partial V_{a0u}} \delta(p_{0i}\bar{u}_i) dS_0 \} - \int_{\partial V_{a0c}} \delta\{x_{ri} - x_{0i}) p_{0i}\} dS_0 + + \int_{\partial V_c} \{h\delta p_n + [(t-t')w_\tau - \Delta u'_\tau] \delta p_\tau + (t-t')|s_\tau| \delta g\} dS \qquad (7b)$$

$$= \sum_{a=1,2} \{ - \int_{V_{a0}} u_i \delta\sigma_{0ij,j} dV_0 + \int_{\partial V_{a0u}} (u_i - \bar{u}_i) \delta p_{0i} dS_0 \} + + \int_{\partial V_c} \{ (\Delta u_n + h) \delta p_n + [(t-t')w_\tau + \Delta u_\tau - \Delta u'_\tau] \delta p_\tau + (t-t')|s_\tau| \delta g \} dS \qquad (7c)$$

x_{ri}: position of particle x_{0i} in the reference state.

These variational inequalities have been derived and verified in Kalker (1986).
We make the following remarks.

a) Both principles are formulated so that the constitutive relation of the bodies are left open. So one can substitute for an elastic body, a viscoelastic body, an elastoplastic body, etc. without having to change the formulation of the contact part of the problem.

b) The principle of virtual work (6) operates on the variation of the displacement field δu_i, which is only restricted by (6a). The variation of the slip magnitude $\delta|s_\tau|$ figures in the integral over δV_c. When expressed in the displacement variation δu_i, the expression $\delta|s_\tau|$ is singular in the three-dimensional case which we consider exclusively when $|s_\tau| = 0$, and this constitutes a grave complication for numerical work. One way of overcoming that difficulty is to regularize the expression $|s_\tau| = \sqrt{s_1{}^2 + s_2{}^2}$ to $\sqrt{s_1{}^2 + s_2{}^2 + \varepsilon}$, where ε is a small enough positive constant. This may lead to numerical difficulties.

c) The principle of complementary virtual work (7) operates on the variation of the stress $\delta\sigma_{0ij}$ and the surface load δp_{0i}. These variations are restricted by (7a). A complication of a finite element method is that the equations of equilibrium $\sigma_{0ij,j} + f_{0i} = 0$ act as a constraint in the interior of the body. On the other hand, a form like $\delta|s_\tau|$ does not occur in the complementary virtual work.

d) We have worked only with three-dimensional bodies in the form of elastic half-spaces. This is a geometric approximation valid when the contact area is a small subregion of the entire surface: its diameter must be small with respect to a typical dimension of the body, and the surface in and near the contact should be smooth. The halfspace approximation was introduced by Hertz (1881) more than a hundred years ago. The process of approximation is as follows: boundary conditions are set up for the real body, which then is approximated by a half-space for the purpose of elasticity calculations. The half-space approximation is remarkably effective. A notably example is the railway wheel-rail system.
The attractiveness of the half-space approximation is that the displacement field due to a point load of arbitrary direction acting at a generic point of the surface of the half-space is explicitly known in the case of linear elastostatics. Then use of the complementary virtual work formulation is clearly indicated, as it carries with it no complications.

4. SPECIALISATION TO LINEAR ELASTICITY

We assume linear elasticity. Then the reference state can be taken equal to the undeformed state, be it that these coinciding states can be time dependent. Also, geometrically, we can let the deformed state coincide with the reference state, so that we only have V_a, ∂V_a, σ_{ij}, u_i, p_i.

The constitutive equations are

$$\left.\begin{array}{l} \sigma_{ij} = E_{ijhk}\, e_{hk}; \qquad e_{hk} = S_{ijhk}\sigma_{ij} \;;\; \sigma_{ij} = \sigma_{ji}; \\ E_{ijhk} = E_{hkij} = E_{khij} \;;\; e_{hk} = \frac{1}{2}(u_{h,k} + u_{k,h}); \;\; e_{hk} = e_{kh}. \\ \sigma_{ij} = E_{ijhk}\, e_{hk} \text{ is a positive definite relationship.} \end{array}\right\} \tag{8}$$

Here,

E_{ijhk} and S_{ijhk} are elastic parameters, which we will take position independent. They may depend on the body number a, however.

e_{hk} is the linearized strain; it is symmetric.

σ_{ij} is the stress; it coincides with the Piola stress. It is symmetric.

Then, the principle of virtual work becomes, when the density $\rho_0 = 0$:

$$\left.\begin{aligned} &\delta Q \le 0 \quad \forall \delta u_i \text{ sub (6a)} \\ &\delta Q = \delta\{ \sum_{a=1,2} [\int_{V_a} \{-\tfrac{1}{2} E_{ijhk}\, u_{i,j}\, u_{h,k} + f_i u_i\} dV + \int_{\partial V_{ap}} \overline{p}_i u_i dS]\} + \\ &-\int_{\partial V_c} (t-t') f p_n\, \delta |s_\tau| dS \end{aligned}\right\} \quad (9)$$

and the principle of complementary virtual work

$$\left.\begin{aligned} &\delta C \ge 0 \quad \forall \delta\sigma_{ij},\ \delta p_i \text{ sub (7a)} \\ &\delta C = \delta\{ \sum_{a=1,2} [\int_{V_a} \tfrac{1}{2} S_{ijhk} \sigma_{ij} \sigma_{hk}\, dV - \int_{\partial V_{au}} p_i \overline{u}_i dS] + \\ &+ \int_{\partial V_c} [h p_n + p_\tau ((t-t') w_\tau - \Delta u'_\tau)] dS\} + \int_{\partial V_c} (t-t') |s_\tau| \delta (f p_n) dS \end{aligned}\right\} \quad (10)$$

5. SURFACE MECHANICAL FORMULATION

We now set the body force f_i equal to zero, and we suppose that in the principle of virtual work (9) and the principle of complementary virtual work (10) the equations of equilibrium are satisfied inside the bodies V_a:

$$\sigma_{ij,j} = 0 \iff E_{ijhk}\, e_{hk,j} = E_{ijhk}\, u_{h,kj} = 0 \quad \text{in } V_a \quad (11)$$

Then the volume integration (9) and (10) may be integrated:

$$\left.\begin{aligned} &\delta Q \le 0 \quad \forall \delta u_i, p_i \text{ sub (6a), (11)} \\ &\delta Q = \delta\{ \sum_{a=1,2} [- \int_{\partial V_{au}} \tfrac{1}{2} p_i u_i\, dS - \int_{\partial V_{ap}} (\tfrac{1}{2} p_i - \overline{p}_i) u_i dS\} + \\ &- \int_{\partial V_c} \{\delta(\tfrac{1}{2} p_i\, \Delta u_i) + (f p_n) \delta |s_\tau|\} dS \end{aligned}\right\} \quad (12)$$

$$\left.\begin{aligned} &\delta C \ge 0 \quad \forall \delta p_i, u_i \text{ sub (7a), (11)} \\ &\delta C = \delta\{ \sum_{a=1,2} [\int_{\partial V_{ap}} \tfrac{1}{2} p_i u_i dS + \int_{\partial V_{au}} p_i (\tfrac{1}{2} u_i - \overline{u}_i)\, dS\} + \\ &+ \int_{\partial V_c} [p_n (\tfrac{1}{2} \Delta u_n + h) + p_\tau ((t-t') w_\tau + \tfrac{1}{2} \Delta u_\tau - \Delta u'_\tau)] dS\} + \\ &+ \int_{\partial V_c} (t-t') |s_\tau| \delta (f p_n) dS \end{aligned}\right\} \quad (13)$$

6. THE CASE OF SYMMETRY

Assume now that the two bodies are geometrically-materially symmetric about a plane through the pot.con. Let this plane be the plane of x and y, and let the axis of z be normal to it, pointing into body 1. The surface tractions acting in the plane $z = 0$ on body a are called (X_a, Y_a, Z_a), and the surface displacements of body a are (u_a, v_a, w_a).

We have that $X_1 = - X_2$, $Y_1 = - Y_2$, $Z_1 = - Z_2$, by Newton's Third Law.

Consider a loading of the bodies which is mirror symmetric with respect to the plane $z = 0$. Then

$$\left.\begin{aligned} &Z_1 = - Z_2 = Z, \quad X_1 = X_2 = 0, \quad Y_1 = Y_2 = 0, \\ &u_1 = u_2, \quad v_1 = v_2, \quad w_1 = - w_2 \end{aligned}\right\} \quad (14)$$

So the displacement differences and tractions become

$$\left.\begin{array}{l}(\Delta u,\Delta v,\Delta w) = (0,0,2w_1)\\ X_1 = X_2 = 0, \quad Y_1 = Y_2 = 0, \quad Z_1 = -Z_2 = Z\end{array}\right\} \qquad (15)$$

Consider now a loading which is mirror antisymmetric with respect to the plane z = 0. Then

$$\left.\begin{array}{lll}X_1 = -X_2 = X, & Y_1 = -Y_2 = Y, & Z_1 = Z_2 = 0\\ u_1 = -u_2, & v_1 = -v_2, & w_1 = w_2\end{array}\right\} \qquad (16)$$

so that $(\Delta u,\Delta v,\Delta w) = (2u_1,2v_1,0)$.
The total field follows by superposition of (15) and (16).

We see that the tangential traction (X,Y) does not influence the normal quantity $\Delta w = \Delta u_n$. So we can calculate the symmetric field first without reference to the tangential traction. Then, p_n is known, and we can calculate the tangential quantities with the traction bound $g = fp_n$, whose variation then vanishes (De Pater (1962)).

We give the surface mechanical formulation for these <u>normal</u> and <u>tangential</u> problems. Indeed, we set $\bar{u}_i = 0$, $\bar{p}_i = 0$ for simplicity, and we substitute the constraints in ∂V_{au} and ∂V_{ap}, viz. $\bar{u}_i = 0$ on ∂V_{au}, and $p_i = 0$ on ∂V_{ap}. We have for the normal problem:

$$\left.\begin{array}{l}\delta Q_n \le 0 \quad \forall \delta u_i, \delta p_i \text{ sub } p_i = 0 \text{ in } \partial V_{ap},\; u_i = 0 \text{ in } \partial V_{au};\; (11),(8);\\ \qquad\qquad e = h + \Delta u_n \ge 0 \text{ in } \partial V_c,\; p_\tau,\; \Delta u_\tau \text{ prescribed in } \partial V_c;\\ \delta Q_n = -\delta\int_{\partial V_c} \frac{1}{2} p_n \Delta u_n dS\\ \delta C_n \ge 0 \quad \forall \delta u_i,\; \delta p_i \text{ sub } p_i = 0 \text{ in } \partial V_{ap},\; u_i = 0 \text{ in } \partial V_{au};\; (11),(8);\\ \qquad\qquad p_n \ge 0 \text{ in } \partial V_c,\; p_\tau,\; \Delta u_\tau \text{ prescribed in } \partial V_c\\ \delta C_n = \delta\int_{\partial V_c} p_n (\frac{1}{2}\Delta u_n + h) dS\end{array}\right\} \qquad (17)$$

and for the tangential problem:

$$\left.\begin{array}{l}\delta Q_\tau \le 0 \quad \forall \delta u_i,\; \delta p_i \text{ sub } u_i = 0 \text{ in } \partial V_{au},\; p_i = 0 \text{ in } \partial V_{ap};\\ \qquad\qquad p_n, \Delta u_n \text{ given in } \partial V_c;\; (11),(8);\\ \delta Q_\tau = -\delta\int_{\partial V_c} [\frac{1}{2} p_\tau \Delta u_\tau + f p_n (t-t') |s_\tau|] dS\\ \delta C_\tau \ge 0 \quad \forall \delta u_i,\; \delta p_i \text{ sub } u_i = 0 \text{ in } \partial V_{au},\; p_i = 0 \text{ in } \partial V_{ap};\; (11),(8)\\ \qquad\qquad |p_\tau| \le f p_n \text{ in } \partial V_c,\; p_n,\; \Delta u_n \text{ given in } \partial V_c\\ \delta C_\tau = \delta\int_{\partial V_c} p_\tau [(t-t') w_\tau + \frac{1}{2}\Delta u_\tau - \Delta u'_\tau] dS\end{array}\right\} \qquad (18)$$

We see that we can define Q_n, C_n, Q_τ, C_τ as functionals of which δQ_n, ..., ..., ... are the variations. The functionals Q_n and Q_τ can be shown strictly concave, and C_n and C_τ can be shown strictly convex. The variational inequalities then imply that at the solution Q_n and Q_τ are maximal, and C_n and C_τ are minimal. Uniqueness of the solution follows at once, and existence follows too if one properly defines the function spaces in which p_i, σ_{ij}, u_i lie.

We note that (12) and (13) cannot be regarded as variations of functionals: (12) and (13) are Galerkin type problems, while (17) and (18) are optimisation problems.
<u>Remark</u>. When the bodies can be regarded as half-spaces for the contact calculation, see sec. 3.d, then the requirement of geometric similarity is automatically satisfied, and the

requirement of material symmetry reduces to the bodies being made of the same material. An example is the contact between wheel and rail in railways, which are both made of steel. Also the requirement of equal materials is usually satisfied in roller bearings, but the half-space approximation is often only marginally satisfied, as one of the dimensions of contact is comparable to a typical dimension of the roller. However, the half-space approximation is so convenient that it often is adopted also in roller bearings.

7. DISCRETISATION

We discretise the problem by dividing the rectangular pot.con. into a regular mesh of equal rectangles S_I, $I = 1,..,M$. On each rectangle it is assumed that a uniform load of intensity p_{Ii} acts. Let x_{Ii} be the center of the rectangle S_I. We assess the displacement difference Δu_{Ii} by considering it in x_{Ii}. We have

$$\Delta u_{Ii} = A_{IiJj}\, p_{Jj}\,, \text{ summation over repeated indices} \tag{19}$$

where A_{IiJj} is called the influence number. In the case of half-spaces, the influence numbers can be calculated analytically, see Kalker (1979). Owing to the regularity of the discretisation net, the number of different influence numbers to be calculated is proportional to the number of meshes N. We usually employ 50 to 100 meshes.

The influence numbers follow from the Boussinesq-Cerruti integral representation, see e.g. Kalker (1979). Consequently the accuracy of the piecewise constant traction distribution which we employ is comparable to that of a continuous, piecewise linear traction distribution. To see this, we employ an analogy: the midpoint rule of numerical integration is equally accurate as the trapezoidal rule. We prefer the piecewise constant traction discretisation, because it leads to simpler expressions than the piecewise linear traction distribution. It may be shown in the case of half-spaces, that the influence numbers A_{I3J3} of the normal contact problem (17) are symmetric, as are the influence numbers $A_{I\alpha J\beta}(\alpha, \beta = 1,2)$ for the tangential problem (18). The remaining influence numbers are antisymmetric. Summarizing,

$$A_{I3J3} = A_{J3I3}\;;\quad A_{I\alpha J\beta} = A_{J\beta I\alpha};\quad A_{I3J\tau} = -A_{J\tau I3} \tag{20}$$

The matrices $(A_{(I3)(J3)})$ and $(A_{(I\alpha)(J\beta)})$, where $(I3),(J3),(I\alpha)$ and $(J\beta)$ are each regarded as a single index rather than as a double one, are positive definite:

$$(A_{(I3)(J3)}) > 0, \quad (A_{(I\alpha)(J\beta)}) > 0, \quad I,J = 1,\dots,M;\ \alpha,\beta = 1,2 \tag{21}$$

The principles of Maximum Work and Minimum Complementary Work become after division by the area of an element, for the normal problem

$$\sup Q_n = -\frac{1}{2} A_{I3J3}\, p_{In} p_{Jn} \quad \text{sub } e_I = h_I + A_{I3J3} p_{Jn} \ge 0 \tag{22a}$$

$$\inf C_n = \frac{1}{2} A_{I3J3}\, p_{In} p_{Jn} + h_I p_{In} \quad \text{sub } p_{In} \ge 0 \tag{22b}$$

We determine the Kuhn-Tucker relations of (22), which are equivalent to the extrematisation owing to convexity properties. We have, by (22a),

$$-A_{I3J3}\, p_{In} = V_I A_{I3J3}\;;\quad V_I \ge 0,\quad V_I e_I = 0,\quad e_I \ge 0 \;\Rightarrow\; p_{In} = V_I \tag{23a}$$

where V_I is a Lagrange multiplier of the constraint.

Also, by (22b), when e_I is the Lagrange multiplier of the constraint,

$$A_{I3J3}\, p_{Jn} + h_I = e_I \geq 0, \quad e_I p_{In} = 0, \quad p_{In} \geq 0 \tag{23b}$$

We see that p_{In} is Lagrange multiplier of the constraint of (22a): Maximum Work, while e_I is Lagrange multiplier of the constraint of (22b): Minimum Complementary Work. We also note that (22a) and (23a) are identical. We denote:

$$\begin{array}{l} \text{The region where } e_I = 0,\ p_{In} \geq 0 \text{ by N; N is called the Contact area} \\ \text{The region where } e_I > 0,\ p_{In} = 0 \text{ by B; B is called the Exterior.} \end{array} \tag{24}$$

Such a region is completely characterised by the indices of the rectangles composing it. These indices form a set, called an index set, which is isomorphic to the geometric region; this is the reason why we will not make a distinction between the geometric region and the index set corresponding to it.

We turn to the tangential problem. First we define the slip:

$$(t-t')s_{I\tau} = (t-t')w_{I\tau} + A_{I\tau J\beta}(p_{J\beta} - p'_{J\beta}) \tag{25}$$

Then we have

$$\sup Q_\tau = -\frac{1}{2} A_{I\alpha J\beta} p_{I\alpha} p_{J\beta} + f p_{I3} |(t-t')s_{I\alpha}| \quad \text{(no auxiliary conditions)} \tag{26a}$$

$$\inf C_\tau = p_{I\alpha}\{(t-t')w_{I\alpha} + A_{I\alpha J\beta}(\tfrac{1}{2} p_{J\beta} - p'_{J\beta})\} \quad \text{sub } |p_{J\tau}| \leq f p_{J3} \tag{26b}$$

(26a) has no Kuhn-Tucker relations, owing to the non-differentiability of $|s_{I\tau}|$, $(\tau = 1,2)$. (26b), on the other hand, possesses Kuhn-Tucker relations. They are:

$$(t-t')w_{I\alpha} + A_{I\alpha J\beta}(p_{J\beta} - p'_{J\beta}) = -V_{(I)} p_{(I)\alpha} / |p_{(I)\tau}| \tag{27}$$

where the notation (I) means that no sum takes place over repeated I.
V_I is the non-negative Lagrange multiplier of the constraint $|p_{I\tau}| \leq f p_{\tau 3}$. The left-hand side of (27) can be identified with $(t-t')s_{I\alpha}$, where $s_{I\alpha}$ is the slip. We conclude that the slip is precisely opposite to the tangential traction. The complementarity relations of V_I can be interpreted as follows, cf. Coulomb's law, eq. (2d,e):

$$\left.\begin{array}{l} \text{When the slip does not vanish } (|s_{\tau\alpha}| > 0) \text{, then } (s_{I\alpha}) \text{ is strictly opposite } (p_{I\alpha}), \\ \text{and } |p_{I\alpha}| = f p_{I3}. \text{ Such I form the area of slip B!} \\ \text{When } |p_{I\alpha}| < f p_{I3} \text{, then } V_I = 0 \text{, and the slip } |s_{I\tau}| \text{ vanishes (area of adhesion N')} \end{array}\right\} \tag{28}$$

8. THE ALGORITHMS

In the algorithms we operate on the Kuhn-Tucker relations. No direct reference is made to the object functions Q_n, C_n, Q_τ, C_τ. The method is a force method, which is induced by the Boussinesq-Cerruti relations (see Love (1926) pp. 192, 243). The algorithms consist of a Worker and a Planner. In the Worker, equations are set up and solved, viz. in the case of normal contact

$$\left.\begin{array}{lll} p_{In} = 0 & I \in & \text{the Exterior } B \\ e_{In} = 0 & I \in & \text{Contact area } N, \end{array}\right\} \qquad (29)$$

and, in the case of tangential contact,

$$\left.\begin{array}{ll} s_{I\tau} = 0 \quad I \in \text{ Adhesion area } N' & \\ \left.\begin{array}{l} |p_{I\tau}| = fp_{I3} \\ (p_{I\tau}) \text{ parallel to } (s_{I\tau}) \end{array}\right\} & I \in \text{ Slip area } B' \end{array}\right\} \qquad (30)$$

We start from certain proposed areas B,N,B',N'; the Planner examines the results of the Worker with a view to the inequalities governing the problem, and on the basis of this examination it either concludes that the solution of the contact problem has been found, or it modifies the current regions.

In the case of the normal contact problem, with Worker (29), the Planner acts as follows:

Action of Planner in Normal Contact Problem (NORM)

1) $\forall I \in$ Contact area N: if $p_{In} < 0$, then I is placed in the Exterior B.
2) If at the end of 1), the Exterior B has grown, Transfer to Worker.
3) $\forall I \in$ the Exterior B: if $e_I \le 0$, then I is placed in Contact area N.
4) If at the end of 3) the contact N has grown, Transfer to Worker.

Now, $p_{In} = 0$, $e_I > 0$ in B;
$p_{In} \ge 0$, $e_I = 0$ in N: we are ready.

In the case of the tangential contact problem, with the Worker (30), the Planner acts as follows:

Action of the Planner of the Tangential Contact Problem (TANG)

1) $\forall I \in$ Adhesion area N': if $|p_{I\tau}| > f_{p\tau 3}$, then I is placed in Slip area B'.
2) If at the end of 1) B' has grown, Transfer to Worker.
3) $\forall I \in$ Slip area B': if $(s_{I\tau})$ is in the same sense as $(p_{I\tau})$, then I is placed in the Adhesion area N'. Note that the Worker has made them parallel!
4) if the Adhesion area has grown at the end of 3) : Transfer to Worker.

Now, $|p_{I\tau}| = fp_{In}$, $(s_{I\tau})$ is opposite $(p_{I\tau})$ in Slip area B';
$|p_{I\tau}| < fp_{In}$, $(s_{I\tau}) = 0$ in Adhesion area N';
see (28): we are ready.

Remarks.

A) We observe that the contact area consists of the areas of Slip and Adhesion: $N = N' \cup B'$. Further, the pot.con. $\{1,\ldots,M\}$ consists of the area of Contact and the Exterior: $\{1,\ldots,M\} = N \cup B$. Of course, N and B are disjoint, as are N' and B'.
B) The algorithms for the normal and the tangential problems are closely related. They are both derived from an algorithm for quadratic programming, which can be proved (Kalker (1988)) to have finite convergence. Although the convergence of the algorithms NORM and TANG has not been rigorously established, they have turned out to be perfectly reliable, and reach their goals after about 1 to 10 activations of the Worker (29) or (30).
C) Note that neither NORM nor TANG make explicit use of the functions C_n and C_τ. This feature is made use of by KOMBI, see below, and in steady state rolling, see sec. 10.

NORM and TANG are fully reliable. However, in principle they can handle only symmetric contact problems. There are two methods to get around this, viz. the Panagiotopoulos (1975) process, and the KOMBI process.

In the Panagiotopoulos process one starts with an application of NORM with $p_{I\tau}^{(0)} \equiv 0$. The result is $p_{In}^{(1)}$. With this $p_{In}^{(1)}$ one calculates the tangential traction $p_{I\tau}^{(1)}$ with the aid of TANG. Then one returns to NORM, and calculates the normal traction $p_{In}^{(2)}$ with $p_{I\tau}^{(1)}$ as the given tangential traction, etc., and one hopes for convergence.

In the 3D case, the Panagiotopoulos process diverges often; in the 2D case its performance is markedly better.
The Panagiotopoulos process consisting of a single application of NORM and TANG is called the Johnson process, see Bentall and Johnson (1967). In the case of symmetry it yields exact results. In the case of asymmetry, the results of the Johnson process are approximate.

The KOMBI process tries to improve on the Panagiotopoulos process by letting the normal and the tangential process interact directly. Such a direct interaction is absent in the Panagiotopoulos process. The KOMBI algorithm also consists of a Worker and a Planner; the Worker solves (29) and (30) as a single problem. The Planner works as follows:

Action of Planner of the Combined Contact Problem (KOMBI)

1) $\forall I \in$ Contact area N: if $p_{In} < 0$, then I is placed in the Exterior B.
   $\forall I \in$ Adhesion area N': if $|p_{I\tau}| > fp_{In}$, then I is placed in Slip area B'.
2) If either B or B' has grown: activate Worker.
3) $\forall I \in$ Slip area B': if $(s_{I\tau})$ is not opposite $(p_{I\tau})$, then I is placed in Adhesion area N'.
4) $\forall I \in$ the Exterior B: if $e_I < 0$ then I is placed in Contact area N. In addition, I is placed in Slip area B'.
5) If either N or N' has grown: activate Worker.
6) READY.

Note that in the case of KOMBI, no object functional C exists.
As yet, KOMBI is not fully reliable. Especially its Worker is far from perfect. Research is still in progress.

9. FEATURES OF THE IMPLEMENTATION

NORM, TANG, and KOMBI are implemented in the program Contact, which is confined to 3D half space elastostatic contact. NORM and TANG are also implemented in the program Laagrol, in which the 2D elastostatic contact of layers is considered, in combination with a Panagiotopoulos process.
The following features of Contact are interesting:

A) TANG and KOMBI work in essentially three modes:
   1. Steady state rolling, see sec. 10.
   2. Transient rolling
   3. Sequences of shifts.

A shift is a contact problem in which two deformable bodies are brought into contact, and then are displaced relative to one another without rolling.

B) In all three algorithms NORM, TANG, and KOMBI one can prescribe one or more components of

11. CONCLUSION

In this paper the weak formulation of contact problems is briefly reviewed. A discretisation in position and time of the weak formulation is introduced. Algorithms for the solution of contact problems are presented, some of which are completely reliable. Contact evolutions are introduced, and some problems concerning them are mentioned; finally, steady state rolling is discussed.

REFERENCES

1 R.H. Bentall, K.L. Johnson (1967), Slip in the rolling contact of two dissimilar elastic rollers, *Int. J. Mech. Sci.* 9, p. 389-404.

2 H. Hertz (1881), see: A.E.H. Love (1926), p. 192 sqq.

3 J.J. Kalker (1979), The computation of three-dimensional rolling contact with dry friction, *Int. J. Num. Meth. Eng.* 14, p. 1293-1307.

4 J.J. Kalker (1988), Contact mechanical algorithms, to appear.

5 A.E.H. Love (1926), A Treatise on the Mathematical Theory of Elasticity, Cambridge U.P.

6 A.D. de Pater (1962), On the reciprocal pressure between two bodies, *Proc. Symp. Rolling Contact Phenomena*, Ed. Bidwell, Elsevier, p. 29-75.

7 P.D. Panagiotopoulos (1975), A non-linear programming approach to the unilateral contact and friction-boundary value problem in the theory of elasticity, *Ing. Arch.* 44, p. 421-432.